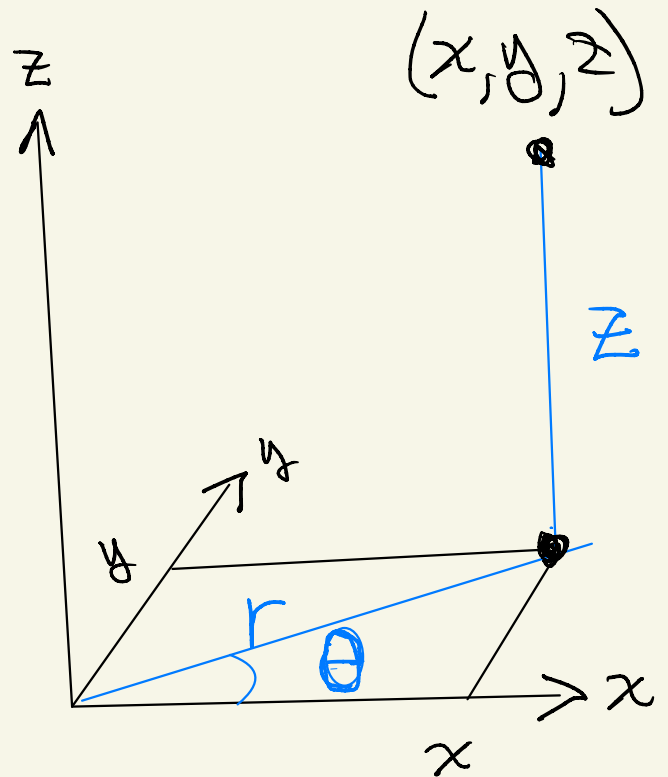


§ 15.6 Spherical/Cylindrical Coords ①
Cylindrical coords are easier but
spherical coords are more important

• Cylindrical coords —

"Uses polar coords for x and y , but keep z unchanged"



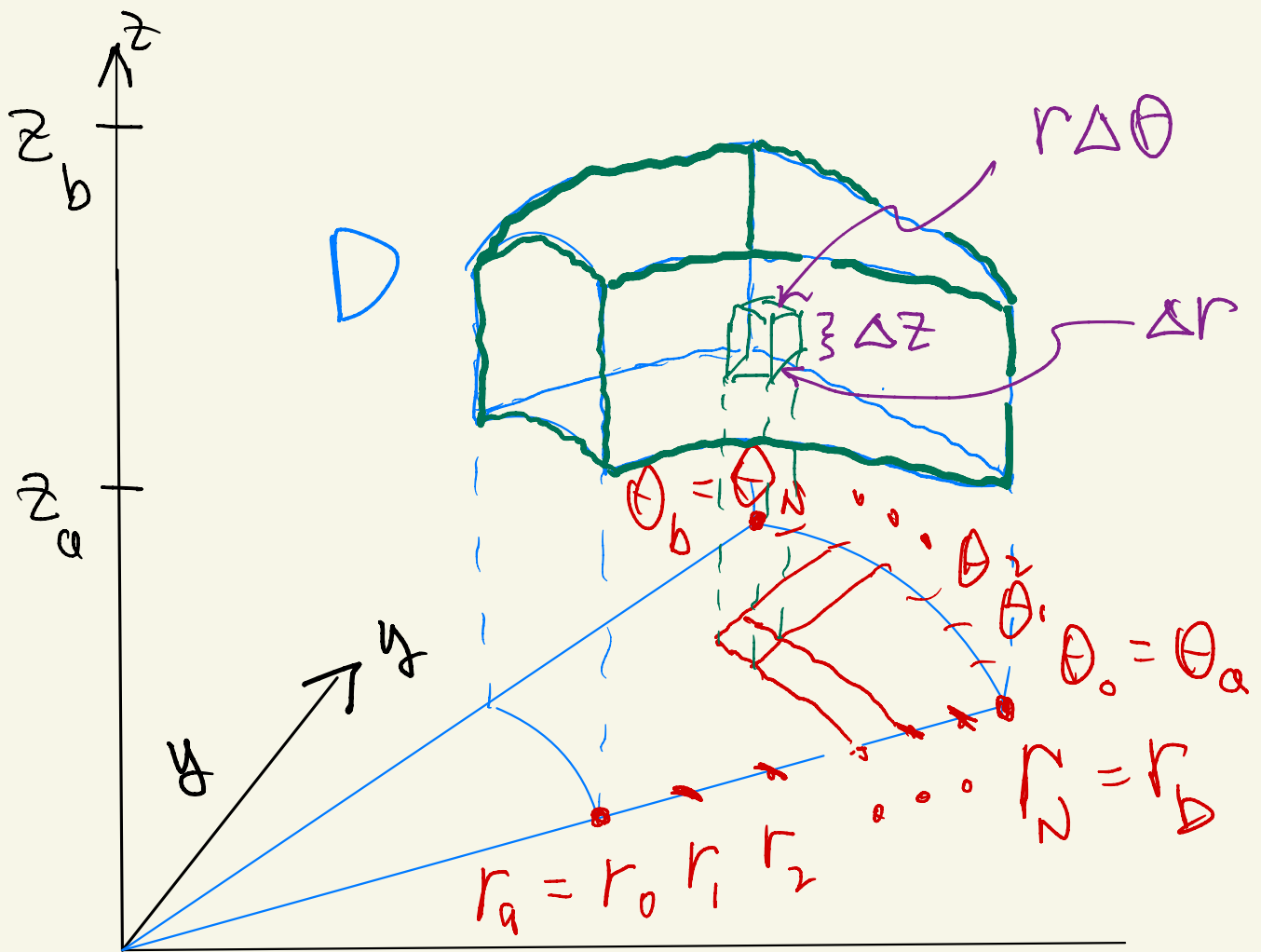
$$x = r \cos \theta$$

$$y = r \sin \theta$$

$$z = z$$

Idea: To change variables of Int.
write Riemann Sum in terms of (r, θ, z)

(2)



- Start with D in (x, y, z) coords
- Discretize in (r, θ, z)

$$\theta_a = \theta_0 < \theta_1 < \dots < \theta_N = \theta_b$$

$$\Delta\theta = \frac{\theta_b - \theta_a}{N}$$

$$r_a = r_0 < r_1 < \dots < r_N = r_b$$

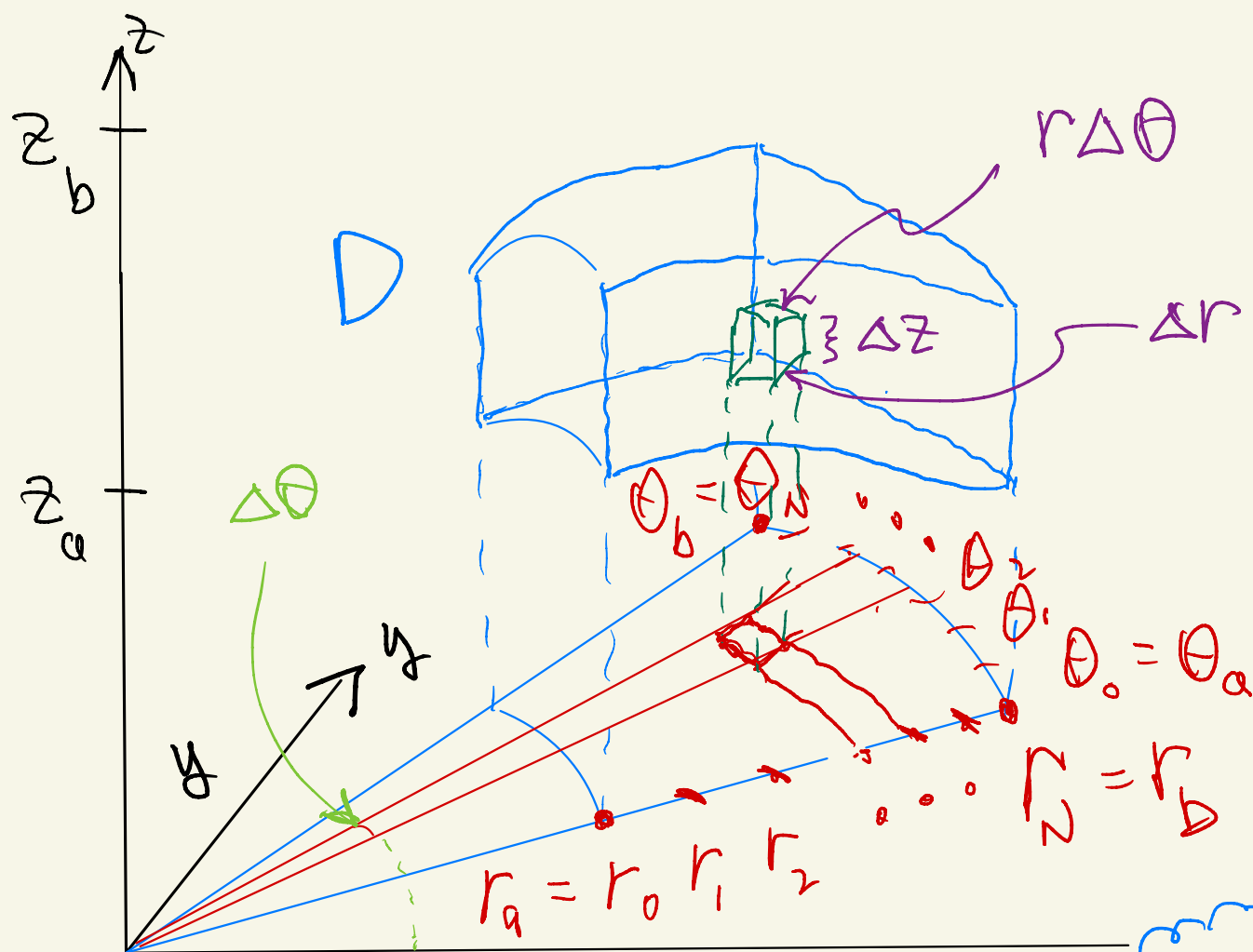
$$\Delta r = \frac{r_b - r_a}{N}$$

$$z_a = z_0 < z_1 < \dots < z_N = z_b$$

$$\Delta z = \frac{z_b - z_a}{N}$$

Express Riemann Sum in (r, θ, z)

(3)



$$I = \iiint_D f(x, y, z) dV$$

Write as limit
of Riemann Sum

$$= \lim_{N \rightarrow \infty} \sum_{(x_i, y_i, z_i) \in D} f(x_i, y_i, z_i) \Delta x \Delta y \Delta z$$

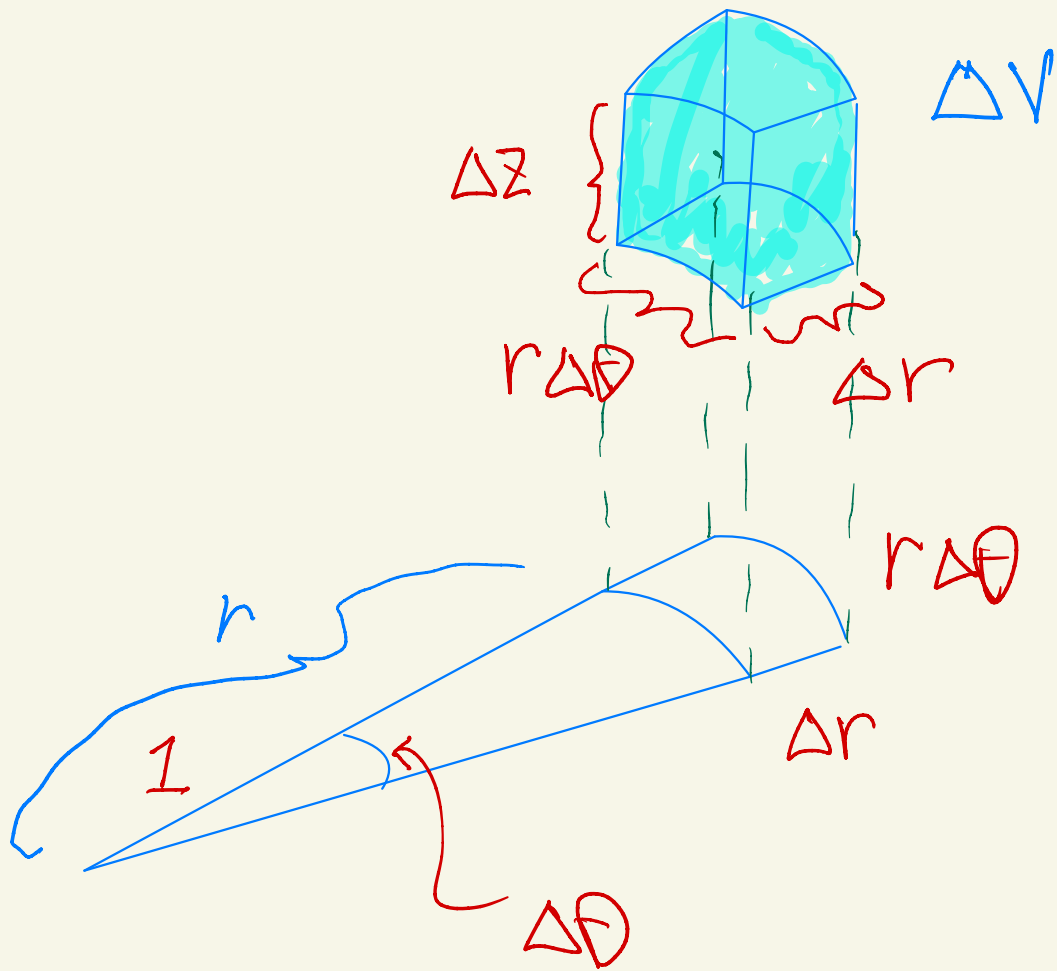
$$= \lim_{N \rightarrow \infty} \sum_{(r_i, \theta_i, z_i) \in D} f(r_i \cos \theta_i, r_i \sin \theta_i, z_i) A \Delta r \Delta \theta \Delta z$$

Amplification
(factor for Vol)

Amplification factor
for volume

To get the amplification factor for Volume, blow up the rectangle $\Delta r \Delta \theta \Delta z \dots$

(5)



Conclude from the geometry:

$$\Delta x \Delta y \Delta z = \Delta V \approx \underset{r}{r} \Delta r \Delta \theta \Delta z$$

Amplification
factor $A = r$

We say: $dx dy dz = dv = r dr d\theta dz$

Conclude:

$$\iiint_D f(x, y, z) \, dV$$

$$= \lim_{N \rightarrow \infty} \sum_{(x_i, y_i, z_i) \in D_{xyz}} f(x_i, y_i, z_i) \Delta x \Delta y \Delta z$$

$$= \lim_{N \rightarrow \infty} \sum_{(r_i, \theta_i, z_i) \in D_{r\theta z}} f(r_i \cos \theta_i, r_i \sin \theta_i, z_i) \underbrace{r_i}_{\substack{w \\ \uparrow \\ A}} \Delta r \Delta \theta \Delta z$$

(Riemann Sum in (r, θ, z))

$$= \iiint_{D_{r\theta z}} f(r \cos \theta, r \sin \theta, z) \underbrace{r \, dr \, d\theta \, dz}_{dA = dx \, dy \, dz}$$

Thm Cylindrical Coordinates

⑦

$$\iiint_{D_{xyz}} f(x, y, z) dV_{xyz}$$

D_{xyz}

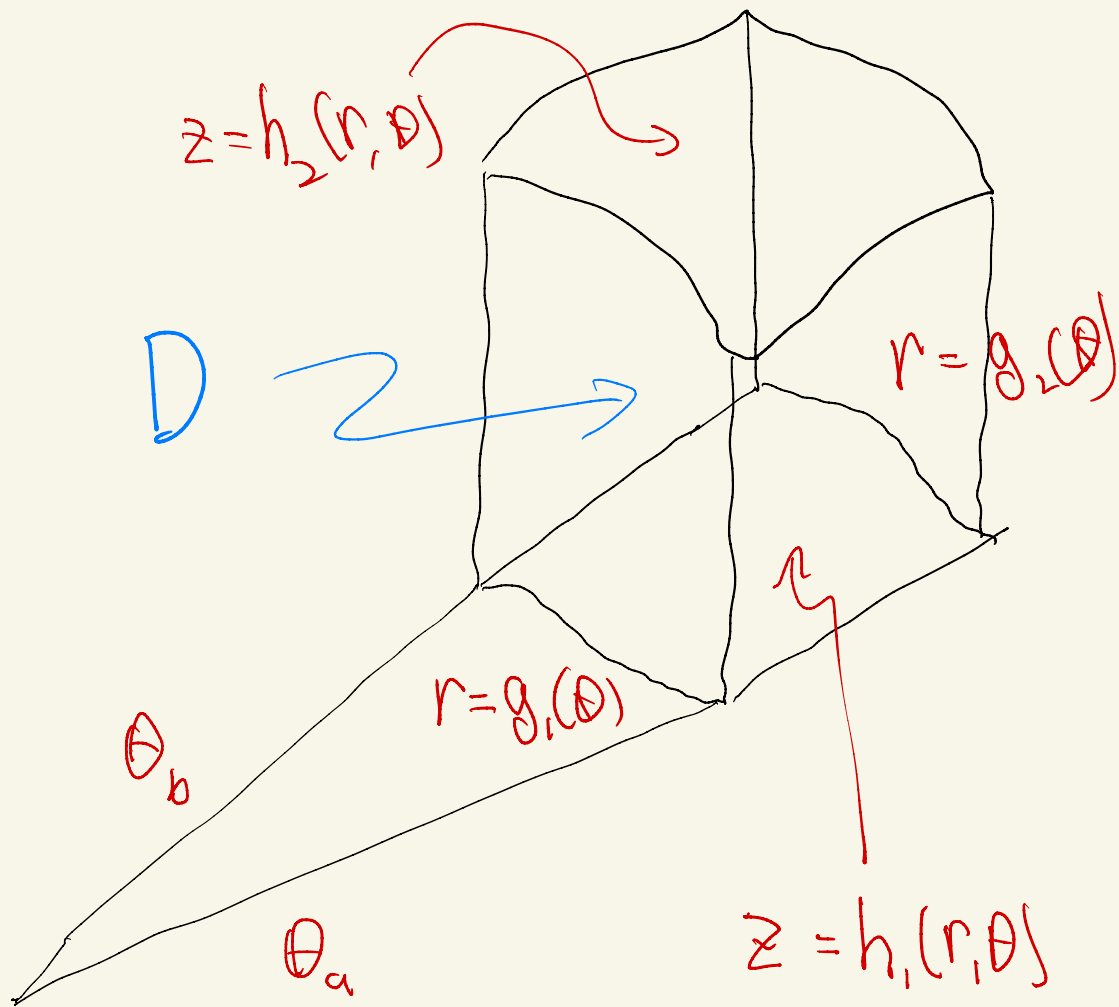
$$= \iiint_{D_{r\theta z}} f(r \cos \theta, r \sin \theta, z) r dV_{r\theta z}$$

$D_{r\theta z}$

If the boundaries line up nicely enough, you can iterate the integral to get an exact value

$\mathbb{R}^3$

9



$$\iiint_D f(x, y, z) dV = \iiint_{D_{r\theta z}} \bar{f}(r, \theta, z) r dr d\theta dz$$

$$= \int_{\theta_a}^{\theta_b} \int_{g_1(\theta)}^{g_2(\theta)} \int_{h_1(r, \theta)}^{h_2(r, \theta)} f(r \cos \theta, r \sin \theta, z) r dz dr d\theta$$

☑ Same idea different geometry for Spherical Coordinates

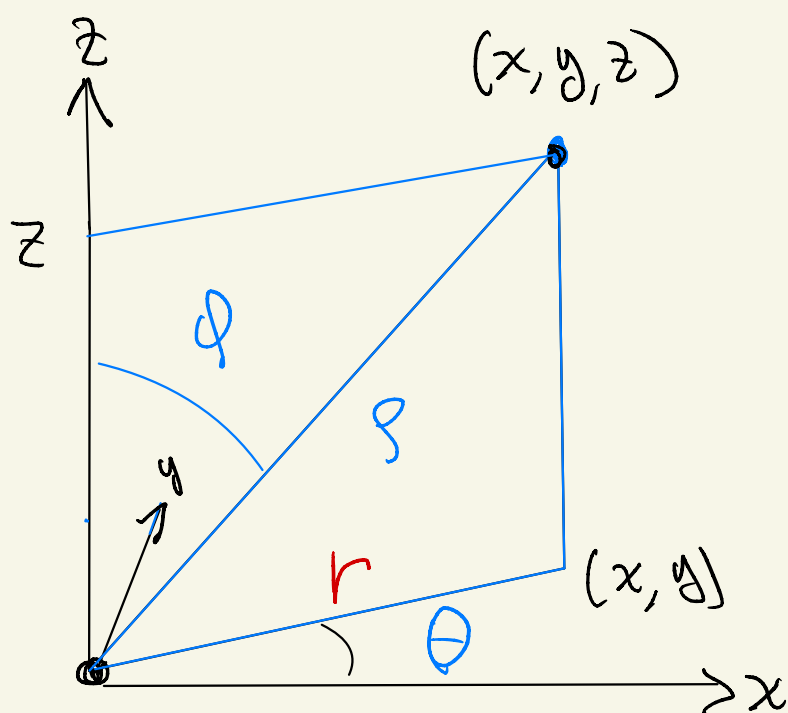
(9)

• Spherical Coordinates:

$$r = \rho \sin \phi$$

$$x = r \cos \theta$$

$$y = r \sin \theta$$



$$x = \rho \cos \theta \sin \phi$$

$$y = \rho \sin \theta \sin \phi$$

$$z = \rho \cos \phi$$

Q: What is the amplification factor for volume?

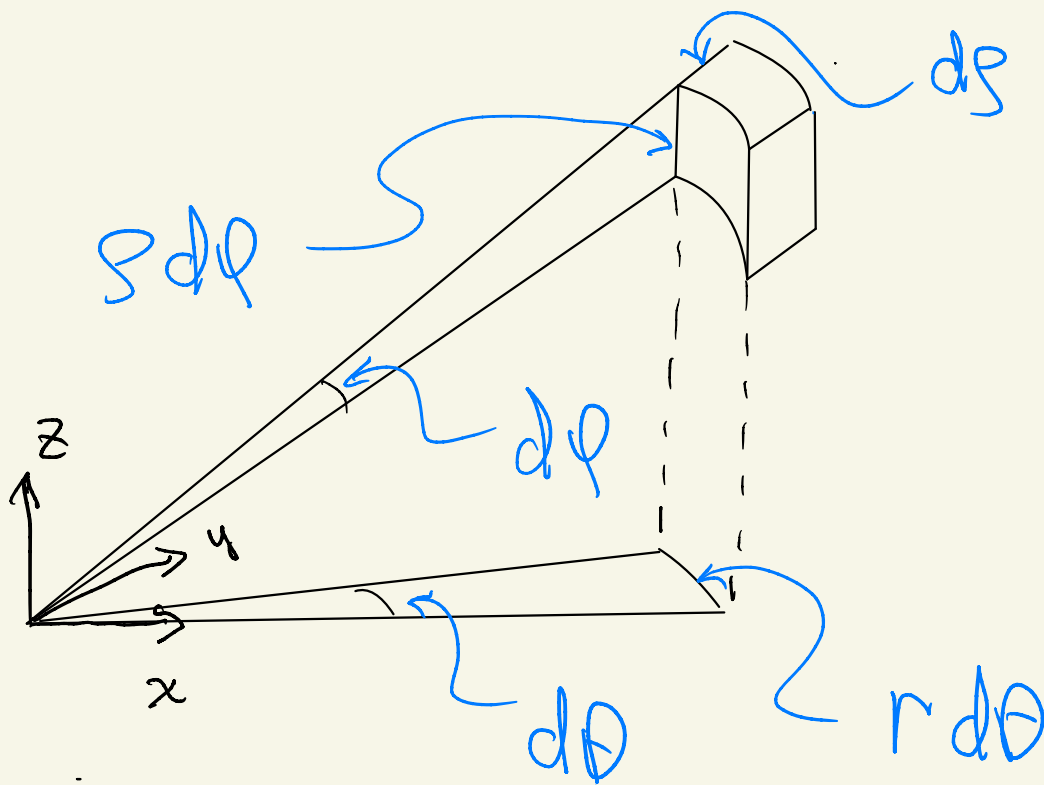
Q: What is A

(10)

$$dx dy dz = A ds d\varphi d\theta$$

Amplification
factor for volume

We get A from geometry:



$$dx dy dz = r s ds d\varphi d\theta = \underbrace{r^2 \sin \varphi}_A ds d\varphi d\theta$$

Theorem: Spherical Coordinates ⑪

$$\iiint_D f(x, y, z) \, dV$$

$$= \iiint_D \tilde{f}(\rho, \varphi, \theta) \underbrace{\rho^2 \sin \varphi}_{A = \rho^2 \sin \varphi} \, d\rho \, d\varphi \, d\theta$$

$$\tilde{f}(\rho, \varphi, \theta) = f(\underbrace{x, y, z})$$

$$(\rho \cos \theta \sin \varphi, \rho \sin \theta \sin \varphi, \rho \cos \varphi)$$

For simple enough functions with ⁽¹²⁾
simple enough geometry you can
get an exact value by iterating
the integral...

(13)

① Example: Find the volume of the ice-cream shaped cone D cut from $\rho \leq 1$ by cone $\phi = \pi/3$

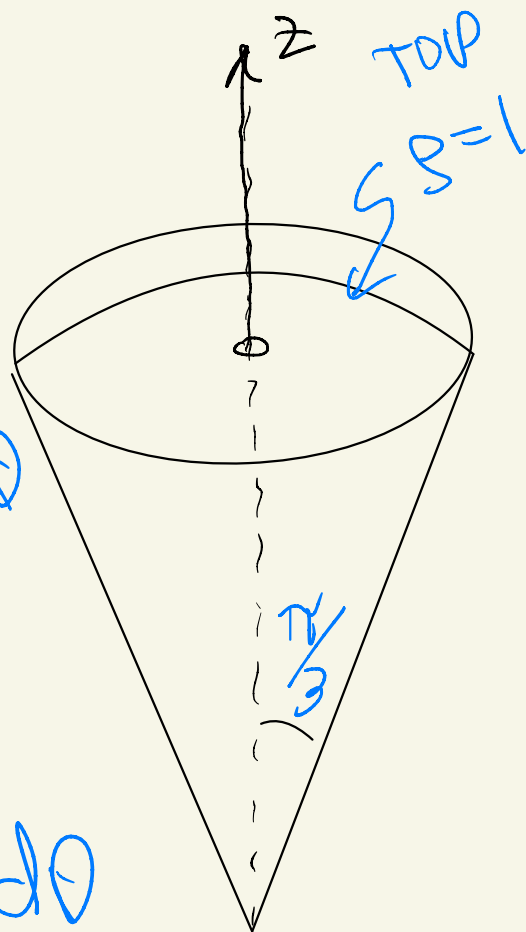
Soln: $V = \iiint_D 1 \cdot dV$

$$= \int_0^{2\pi} \int_0^{\pi/3} \int_0^1 \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta$$

$$= \int_0^{2\pi} \int_0^{\pi/3} \sin \phi \left[\frac{\rho^3}{3} \right]_0^1 d\phi \, d\theta$$

$$= \frac{1}{3} \int_0^{2\pi} \underbrace{-\cos \phi \Big|_0^{\pi/3}}_{-\cos \frac{\pi}{3} + \cos 0} d\theta = \frac{1}{3} \cdot \frac{1}{2} \cdot 2\pi = \boxed{\frac{\pi}{3}}$$

$$-\cos \frac{\pi}{3} + \cos 0 = -\frac{1}{2} + 1 = \frac{1}{2}$$

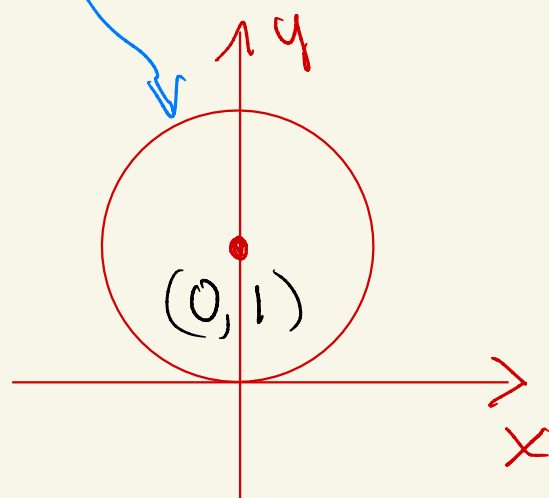
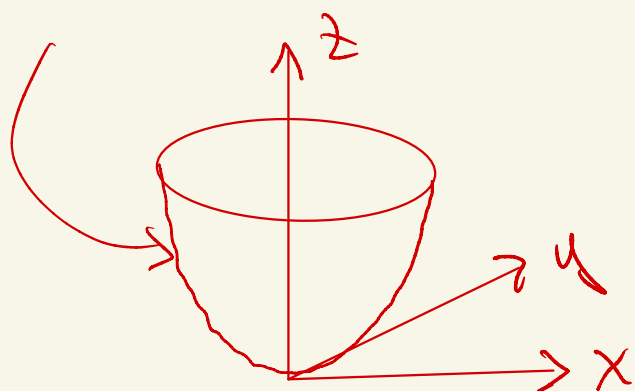


(2) Iterate (but don't evaluate) (14)

the integral $I = \iiint_D f(r, \theta, z) dv$

where D is the region bdd
below by plane $z=0$, laterally
by the circular cylinder
 $x^2 + (y-1)^2 = 1$ and above by
paraboloid $z = x^2 + y^2$

$x^2 + (y-1)^2 = 1$ circle center
 $(x, y) = (0, 1)$
 $z = x^2 + y^2$ paraboloid

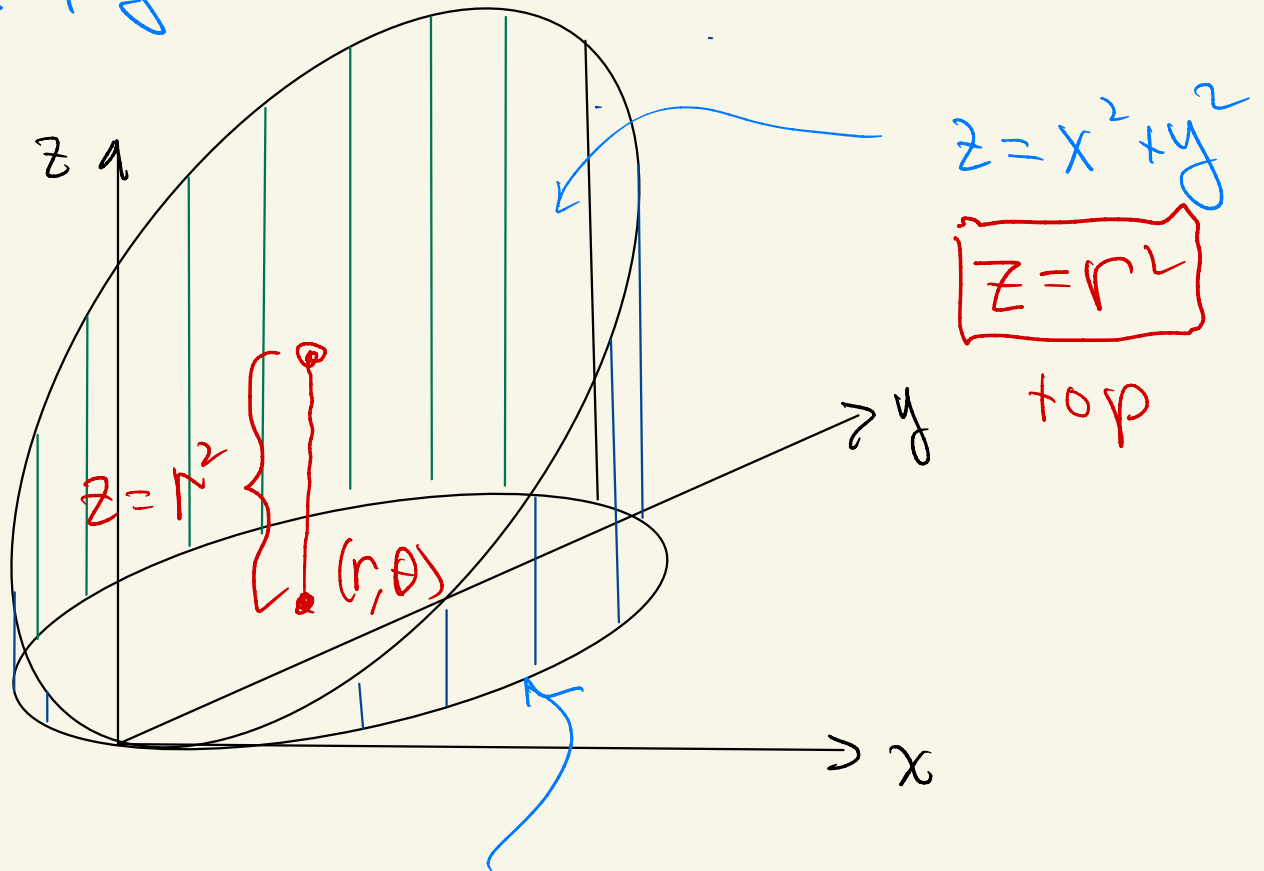


$$x^2 + (y-1)^2 = 1$$

$$z = x^2 + y^2$$

Graph:

(15)



$$x^2 + y^2 - 2y + 1 = 1$$

$$r^2 = 2y = 2r \cos \theta$$

$$r = 2 \cos \theta \quad 0 \leq \theta \leq \pi$$

$$I = \int_0^\pi \int_0^{2\cos\theta} \int_0^{r^2} f(r, \theta, z) r \, dz \, dr \, d\theta$$

§ 16.7 General coordinate transformations —

(16)

- We used the geometry of polar/cylindrical/spherical coordinate transformations to get the amplification factor for area/volume

Polar: $dx dy = r dr d\theta$

Cylindrical: $dx dy dz = r dr d\theta dz$

Spherical: $dx dy dz = r^2 \sin\phi dr d\phi d\theta$

- How does it work in general?

2-D

$$x = g(u, v)$$

$$y = h(u, v)$$

or

$$x = g(u, v, w)$$

$$y = h(u, v, w)$$

$$z = k(u, v, w)$$

- Compare with the 1-D substitution principle of Math 21B

(17)

$$I = \int_a^b f(x) dx = \int_{u_a}^{u_b} f(g(u)) g'(u) du$$

$b = g(u_b)$
 $a = g(u_a)$ A

$$x = g(u)$$

$$dx = g'(u) du$$

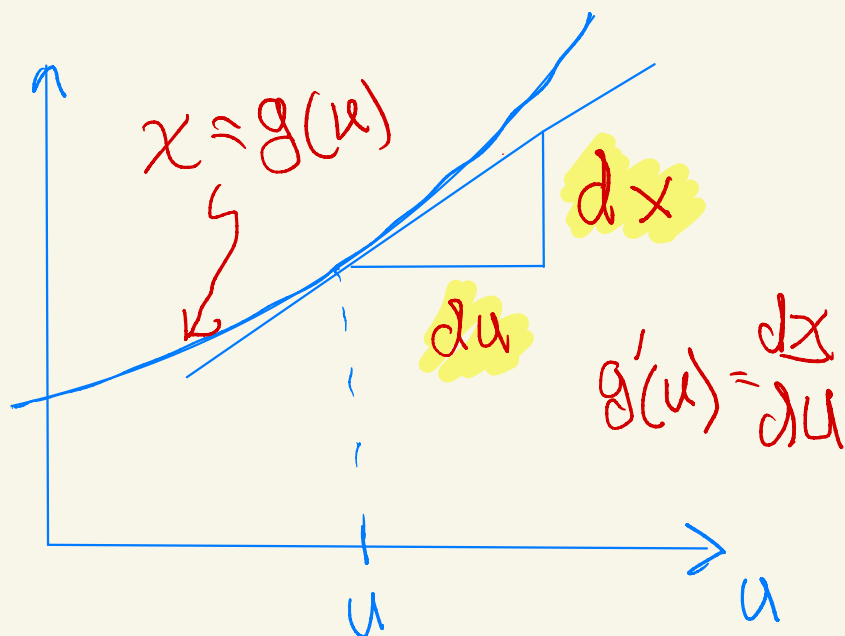
Amplification factor for length
 $A = g'(u)$

Picture:

$$m = g'(u) = \frac{dx}{du}$$

$$dx = g'(u) du$$

A



- Consider 2-dimensions



$$I = \iint_{R_{xy}} f(x, y) dx dy$$

$$\begin{aligned} x &= g(u, v) \\ y &= h(u, v) \end{aligned}$$

Theorem :

$$\iint_{R_{xy}} f(x, y) dx dy$$

$$= \iint_{R_{uv}} f(g(u, v), h(u, v)) A du dv$$

$$\begin{aligned} A &= \det \begin{vmatrix} g_u & g_v \\ h_u & h_v \end{vmatrix} = \det \begin{vmatrix} -\nabla g - \\ -\nabla h - \end{vmatrix} \\ &= g_u h_v - g_v h_u \end{aligned}$$

Ex Check Polar Coordinates:

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$$x = r \cos \theta = g(r, \theta) \quad u = r$$

$$y = r \sin \theta = h(r, \theta) \quad v = \theta$$

$$\nabla g = \left(\frac{\partial g}{\partial r}, \frac{\partial g}{\partial \theta} \right) = (\cos \theta, -r \sin \theta)$$

$$\nabla h = \left(\frac{\partial h}{\partial r}, \frac{\partial h}{\partial \theta} \right) = (\sin \theta, r \cos \theta)$$

$$A = \det \begin{vmatrix} -\nabla g \\ -\nabla h \end{vmatrix} = \det \begin{vmatrix} \cos \theta & -r \sin \theta \\ \sin \theta & r \cos \theta \end{vmatrix}$$

$$= \overset{g_u}{(\cos \theta)} \overset{h_v}{(r \cos \theta)} - \overset{g_v}{(-r \sin \theta)} \overset{h_u}{(\sin \theta)}$$

$$= r \cos^2 \theta + r \sin^2 \theta = r(\cos^2 \theta + \sin^2 \theta)$$

$$= r \quad \checkmark$$

Summary

$$\bar{f}(u,v) = f(g(u,v), h(u,v)) \quad (20)$$

2-D: $\iint_{R_{xy}} f(x,y) \, dx \, dy = \iint_{R_{uv}} \bar{f}(u,v) \begin{vmatrix} -\nabla g \\ -\nabla h \end{vmatrix} du \, dv$

$dA_{xy} \uparrow R_{uv}$ dA_{uv}

$x = g(u,v)$
 $y = h(u,v)$

$\det \begin{vmatrix} g_u & g_v \\ h_u & h_v \end{vmatrix}$

3-D: $\iiint_{D_{xyz}} f(x,y,z) \, dA_{xyz} = \iiint_{D_{uvw}} \bar{f}(u,v,w) \begin{vmatrix} -\nabla g \\ -\nabla h \\ -\nabla k \end{vmatrix} dA_{uvw}$

$D_{xyz} \uparrow D_{uvw}$

$x = g(u,v,w)$
 $y = h(u,v,w)$
 $z = k(u,v,w)$

$\det \begin{vmatrix} g_u & g_v & g_w \\ h_u & h_v & h_w \\ k_u & k_v & k_w \end{vmatrix}$